Electromagnetic Structure of the Nucleon in Local-Field Theory*

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In analogy to the dispersion-relation method for scattering, the description of nucleon electromagnetic structure by local-field theory is discussed in terms of mass-spectral representations for the form factors. The existence of such representations is made plausible although not proved, and it is shown that the spectral distribution functions are related to scattering amplitudes on the mass shell but sometimes in a nonphysical region. It is argued that the main contributor to the magnetic moment structure in the spectral distribution must be the two-pion state, and an attempt is made to evaluate this contribution in terms of the known behavior of pion-nucleon scattering. A semiquantitative calculation yields results in reasonable agreement with experiment.

It is emphasized that the large observed charge radius of the proton does not imply the dominance of the two-pion state in the charge structure. Thus it is not impossible that higher mass configurations supply the isotopic scalar charge needed to explain the small neutron-electron interaction.

I. INTRODUCTION

THEORETICAL calculations of the electromagnetic properties of the nucleon have been carried out for many years within the framework of local-field theory, but mainly by perturbation techniques[1] of dubious validity. Recently the use of dispersion relations in the problem of pion-nucleon scattering[2] and photopion production[3] has shown that local-field theory is capable of some quantitative correlation of physical phenomena even when the perturbation method fails. It is the purpose of this paper to attempt to apply the kind of relations that have successfully correlated experiments involving low-energy pions to the problem of the nucleon electromagnetic form factors. To the extent at least that the electromagnetic structure of the nucleon is determined by virtual pions of sub-Bev frequencies such a program should be enlightening, even though in the end local theories in the strict sense may be abandoned.

1.—There are at least three reasons for believing that the anomalous magnetic-moments structure of the nucleon is dominated by low-frequency virtual pions:

(*a*) The anomalous moment is almost entirely a vector in isotopic spin, i.e., the anomalous moments of neutron and proton are nearly equal in magnitude, with opposite signs. This situation prevails not only for the static moments but up to frequencies at which the moments have fallen to about $\frac{1}{3}$ of their static values.[4] It will be explained below that the π^+, π^- pair, the virtual configuration of lowest energy contributing to the nucleon electromagnetic structure, is a vector in isotopic spin space. It is of course possible for a combination of virtual effects other than pion pairs to produce an almost purely vector moment, but such a circumstance must be regarded as unlikely.

(*b*) The sign and the approximate magnitude of the anomalous moments are correctly given by the cutoff model of the Yukawa theory.[5] This model is normalized to the same low-frequency limits as the local theory, but neglects nucleon recoil as well as antinucleons and strange particles and excludes virtual pions of energy higher than about 1 Bev.

(*c*) The measured mean square radius of the magnetic-moment distribution[4] corresponds to the wavelength of a pion of about $\frac{1}{3}$ Bev.

In contrast to the anomalous magnetic moment, it is experimentally clear that the charge structure of the nucleon is *not* dominated by π^+, π^- pairs. The decisive fact here is the extremely small second radial moment of the neutron charge distribution as compared with that for the proton, which is at least ten times as large.[6] Thus the charge density is certainly not an isotopic vector.

2.—Before going into the details it is perhaps advisable to outline the approach to be used. It is well known that the linear interaction of nucleons with the electromagnetic field can be expressed in terms of four real scalar functions of q^2, the square of the energy-momentum-transfer four-vector.[6] We shall label these

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[1] References to most of the published perturbation calculations may be found in the paper by B. Fried, Phys. Rev. **88**, 1142 (1952), who gives the formulas for the neutron to lowest order in the pion-nucleon coupling constant. A numerical evaluation of the form factors is given by M. Rosenbluth, Phys. Rev. **79**, 615 (1950).

[2] References here may be found in the recent paper by Chew, Goldberger, Low, and Nambu, Phys. Rev. **106**, 1337 (1957).

[3] Chew, Goldberger, Low, and Nambu, Phys. Rev. **106**, 1345 (1957).

[4] E. E. Chambers and R. Hofstadter, Phys. Rev. **103**, 1454 (1956), and R. Hofstadter, in *Proceedings of the Seventh Annual Rochester Conference on High-Energy Nuclear Physics* (Interscience Publishers, Inc., New York, 1957).

[5] H. Miyazawa, Phys. Rev. **101**, 1564 (1955).

[6] For a recent review of experimental knowledge about the electromagnetic structure of the nucleon, see Lévy, Ravenhall, and Yennie, Revs. Modern Phys. **29**, 144 (1957). This article also discusses those theoretical features of the problem which follow from invariance considerations.

functions $G_1{}^S(q^2)$, $G_1{}^V(q^2)$, $G_2{}^S(q^2)$, $G_2{}^V(q^2)$, where the index 1 goes with the part of the interaction proportional to $\gamma_\mu A_\mu$ (the "charge") and the index 2 goes with the part of the interaction proportional to $\sigma_{\mu\nu}A_\mu q_\nu$ (the "magnetic moment").[6] The superscripts S and V refer to the isotopic character of the interaction, scalar or vector, the normalization being specified by the relations

$$G_1{}^S(0)+G_1{}^V(0)=e,\quad G_1{}^S(0)-G_1{}^V(0)=0, \tag{2.1}$$

$$G_2{}^S(0)+G_2{}^V(0)=\mu_p,\quad G_2{}^S(0)-G_2{}^V(0)=\mu_n, \tag{2.2}$$

where e is the proton charge and μ_p and μ_n the proton and neutron static anomalous magnetic moments, respectively. The conventional form factors[6] are given by the ratio of the appropriate $G(q^2)$ to the value at $q^2=0$. Thus in our notation the proton form factors are

$$F_{1,2}{}^p(q^2)=\frac{G_{1,2}{}^S(q^2)+G_{1,2}{}^V(q^2)}{G_{1,2}{}^S(0)+G_{1,2}{}^V(0)}.$$

Our approach is to be based on mass spectral representations of the type

$$G_1{}^S(q^2)=\frac{e}{2}-\frac{q^2}{\pi}\int_{(3m_\pi)^2}^\infty dm^2\frac{g_1{}^S(m^2)}{m^2(m^2+q^2)}, \tag{2.3}$$

$$G_1{}^V(q^2)=\frac{e}{2}-\frac{q^2}{\pi}\int_{(2m_\pi)^2}^\infty dm^2\frac{g_1{}^V(m^2)}{m^2(m^2+q^2)}, \tag{2.4}$$

$$G_2{}^S(q^2)=\frac{1}{\pi}\int_{(3m_\pi)^2}^\infty dm^2\frac{g_2{}^S(m^2)}{m^2+q^2}, \tag{2.5}$$

$$G_2{}^V(q^2)=\frac{1}{\pi}\int_{(2m_\pi)^2}^\infty dm_2\frac{g_2{}^V(m^2)}{m^2+q^2}, \tag{2.6}$$

which have been suggested by a number of authors.[7–9] The four real weight functions $g_{1,2}{}^{S,V}(m^2)$ may be nonzero for m equal to the mass of any system strongly coupled to the nucleon which at the same time can be created by the electromagnetic field. The lightest such isotopic vector system is the π^+, π^- pair, while the three-pion π^+, π^-, π^0 system is the lightest isotopic scalar; hence the thresholds at $(2m_\pi)^2$ and $(3m_\pi)^2$. It will be shown in Sec. III that, in general, systems of even numbers of pions contribute only to the isotopic vector charge and magnetic moment while odd numbers of pions give purely isotopic scalar contributions. Of a mass comparable to six pions is the K^+, K^- pair, and eventually of course one comes to the baryon pairs, starting with the nucleon-antinucleon system. From a practical standpoint one must hope that in the mass spectra the contributions from the simplest systems are the most important.

The derivation of the representations (2.3)–(2.6) to be given in Sec. II presupposes that $G_1(Z)/Z$ and $G_2(Z)$ approach zero for large Z. Actually, it may be inferred from the work of Lehmann, Symanzik, and Zimmermann[10] that $G_1(Z)$ approaches zero also. In that case one may write a relation of the form

$$G_1{}^{S,V}(q^2)=\frac{1}{\pi}\int dm^2\frac{g_1{}^{S,V}(m^2)}{m^2+q^2}, \tag{2.7}$$

with the restriction on $g_1{}^{S,V}$ implied by Eq. (2.1). The convergence, however, is achieved only because of electromagnetic damping, which sets in for extremely large $q^2\gtrsim M^2e^{137}$, while according to perturbation theory[1] the functions G_1 behave logarithmically for large q^2 in the range $M^2e^{137}\gg q^2\gg M^2$. It is possible that an exact solution of the pion-nucleon field theory would lead to functions G_1 which tend to zero even without electromagnetic damping. At present, however, we cannot feel at all confident of such a circumstance, so we prefer to use Eqs. (2.3) and (2.4) to avoid a large contribution from the uncertain regions. The anomalous magnetic-moment distribution, on the other hand, for reasons which are essentially dimensional, is definitely expected to approach zero for $q^2\gg M^2$ with or without electromagnetic damping. Thus for practical purposes we are confronted by a difference between the charge and magnetic moment distributions.

3.—Often it seems appropriate to discuss the nucleon electromagnetic structure in configuration-space language, and to that end one conventionally introduces three-dimensional Fourier transforms of the functions $G_{1,2}{}^{S,V}$:

$$\rho^{S,V}(r)=\frac{1}{(2\pi)^3}\int d\mathbf{p}\, e^{i\mathbf{p}\cdot\mathbf{r}}G_1{}^{S',V}(p^2), \tag{3.1}$$

$$\mathfrak{M}^{S,V}(r)=\frac{1}{(2\pi)^3}\int d\mathbf{p}\, e^{i\mathbf{p}\cdot\mathbf{r}}G_2{}^{S,V}(p^2). \tag{3.2}$$

Although the configuration-space functions ρ and $\mathfrak{M}$ have no precise physical meaning they correspond roughly to charge and anomalous magnetic-moment densities, respectively. Substituting Eqs. (2.5) and (2.6) into (3.2), we have

$$\mathfrak{M}^{S,V}(r)=\frac{1}{2\pi^2}\int dm^2\, g_2{}^{S,V}(m^2)\frac{e^{-mr}}{r}, \tag{3.3}$$

which shows that in the spectral decomposition of the magnetic moment the contribution of a particular mass value m has a "range" $\sim 1/m$. Thus the lightest masses that contribute to $g_2(m^2)$ give rise to the longest-range structure.

A quantity often used to characterize the size of the nucleon is the "mean square radius of the anomalous magnetic moment,"[6] that is (suppressing the super-

[7] Y. Nambu, Nuovo cimento **6**, 1064 (1957).
[8] V. Glaser and B. Jaksic, Nuovo cimento **5**, 1197 (1957).
[9] M. Gell-Mann (private communication).

[10] Lehmann, Symanzik, and Zimmermann, Nuovo cimento **2**, 425 (1955).

scripts S and V),

$$\langle r_m^2\rangle_{\text{Av}}=\int d\mathbf{r}\, r^2\mathfrak{M}(r)\bigg/\int d\mathbf{r}\,\mathfrak{M}(r), \tag{3.4}$$

which is easily shown to be related to the logarithmic derivative of $G_2(q^2)$ at $q^2=0$:

$$\tfrac{1}{6}\langle r_m^2\rangle_{\text{Av}}=-\frac{1}{G_2(0)}\left[\frac{dG_2(q^2)}{dq^2}\right]_{q^2=0}, \tag{3.5}$$

or

$$\tfrac{1}{6}\langle r_m^2\rangle_{\text{Av}}=\int dm^2\frac{g_2(m^2)}{m^4}\bigg/\int dm^2\frac{g_2(m^2)}{m^2}. \tag{3.6}$$

Thus the mean square radius is related to some average mass in the weight function g_2/m^2,

$$\tfrac{1}{6}\langle r_m^2\rangle_{\text{Av}}=\langle m^{-2}\rangle_{\text{Av}}, \tag{3.7}$$

a notion which is useful if the spectral distribution is predominantly of one sign. Actual calculation, as will be seen in Sec. V, shows no tendency for g_2 to oscillate, although it has not been proved that a change of sign is impossible. Taking the measured root-mean-square radius of the anomalous (vector) nucleon magnetic moment,[4] one finds a corresponding average mass of $5m_\pi$, which, if divided between two particles, would give each an average total energy of 2.5 m_π. This low average energy suggests, as mentioned above, that virtual K particles and baryons play only a small role in the determination of the anomalous magnetic moment.

4.—Because of the uncertain behavior of $G_1(q^2)$ at infinity, there may not exist a useful connection between the second radial moment of the charge distribution and an average virtual mass. Going through the same manipulations as above but using Eqs. (2.3) and (2.4) rather than (2.5) and (2.6), one finds for the mean square radius of the (scalar or vector) charge the formulas

$$\tfrac{1}{6}\langle(r_\rho^2)\rangle_{\text{Av}}{}^S=\frac{2}{\pi e}\int_{(3m_\pi)^2}^{\infty}\frac{g_1{}^S(m^2)}{m^4}dm^2, \tag{4.1}$$

$$\tfrac{1}{6}\langle(r_\rho^2)\rangle_{\text{Av}}{}^V=\frac{2}{\pi e}\int_{(2m_\pi)^2}^{\infty}\frac{g_1{}^V(m^2)}{m^4}dm^2. \tag{4.2}$$

Often the statement is made that because the lowest-mass intermediate state the π^+, π^- pair, contributes to the vector charge but not to the scalar the latter should have a much smaller mean square radius than the former. Such reasoning, however, is tacitly based on the assumption that a formula of the type of (3.7) holds for the charge radius as well as for that of the magnetic moment. Formulas (4.1) and (4.2) in themselves imply nothing about either the relative or the absolute magnitudes of the second radial moments of the scalar and vector charge distributions.

The experimental fact that the scalar and vector second radial moments of the charge are almost equal means, of course, that configurations more complicated than the π^+, π^- pair are important. Why this should not also be true for the magnetic moment we must say at once we do not understand. It is, however, fortunate that at least part of the problem of the nucleon electromagnetic structure may be tractable.

5.—In our present state of knowledge an attempt at a specific evaluation of the weight functions $g_{1,2}{}^{S,V}(m^2)$ must be confined to the two-pion contribution, and even here we have not succeeded in formulating a reliable method of calculation. We shall show that the two-pion part of the weight function is proportional to the charge-exchange pion-nucleon scattering amplitude, but at a negative value for the square of the momentum transfer. An extension of the physical scattering amplitude is thus required, which we attempt to carry out by means of dispersion relations combined with Legendre polynomials. If integrals are cut off and an expansion is made in inverse powers of the nucleon mass the results of the static model[5] can be reproduced. Without a cutoff we are unable to make a definite calculation, but arguments will be given to support the belief that the local theory, properly evaluated, will be in agreement with the observations.

In Sec. II we discuss and to some extent justify the representations (2.2) to (2.6). Section III deals with general properties of the various intermediate-state contributions to the weight functions $g_{1,2}{}^{S,V}(m^2)$, and in Secs. IV and V we concentrate on the two-pion intermediate state. In Sec. VI our findings are summarized.

II. THE MASS-SPECTRAL REPRESENTATIONS

6.—Recently Bogoliubov, Medvedev, and Polivanov[11] and others[12] derived dispersion relations for meson-nucleon scattering from the causal nature of a local-field theory. In this section we shall show that the electromagnetic structure factor satisfies requirements that are analogous to the properties of the meson-nucleon scattering amplitude. We therefore infer that it has a spectral representation similar to the dispersion relation for the scattering amplitude. Our discussion closely follows that in reference 11.

We shall write the form factor for the emission of a virtual four-vector quantum with momentum $q_\mu(0\leq q^2)$,

$$\bar{u}(p's')F_\mu(p',q;p)u(p,s), \tag{6.1}$$

where the nucleon makes a transition from the state with momentum p, spin and isotopic spin s, to the state p', s'; u and $\bar{u}$ are the usual normalized spinors. The index s will be suppressed where no loss of clarity results. If the field operator $A_\mu(x)$ for the virtual electromagnetic field is introduced in addition to the

[11] Bogoliubov Medvedev, and Polivanov, Institute for Advanced Study Notes, Princeton, New Jersey, 1956 (unpublished).

[12] Bremermann, Oehme, and Taylor, Phys. Rev. **109**, 2178 (1958).

nucleon operators $\bar{\psi}(x)$ and $\psi(x)$, we can consider the function in Eq. (6.1) as an S-matrix element to which the reduction formulas[11] can be applied:

$$\bar{u}(p's')F_\mu(p',q;p)u(p,s)=(p's',q_\mu|S|ps)$$
$$=\frac{1}{(2\pi)^3}\int d^4x\,d^4y\,e^{-iqx}e^{ipy}\times\left(p's'\left|\frac{\delta^2S}{\delta A_\mu(x)\delta\psi(y)}S^\dagger\right|0\right)u(ps)$$
$$=-\frac{1}{(2\pi)^3}\int d^4x\,d^4y\,e^{-iqx}e^{ipy}\times(p's'|T(j_\mu(x)\bar{\eta}_\beta(y)|0)u(p,s),\quad(6.2)$$

plus a possible local contribution to the integrand when $x=y$. Here the currents are

$$j_\mu(x)=i\frac{\delta S}{\delta A_\mu(x)}S^\dagger,\quad \bar{\eta}_\beta(y)=i\frac{\delta S}{\delta\psi_\beta(y)}S^\dagger.\quad(6.3)$$

In the final step of Eq. (6.2), the causality conditions have been used in the form

$$\delta j_\mu(x)/\delta\psi_\beta(y)=0,\quad x_0>y_0\quad\text{or}\quad(x-y^2)>0;$$
$$\delta\bar{\eta}_\beta(y)/\delta A_\mu(x)=0,\quad x_0<y_0\quad\text{or}\quad(x-y)^2>0.\quad(6.4)$$

We may now define the causal function $S^{(c)}$ and a set of related covariant functions,

$$(p'|T[j_\mu(x),\bar{\eta}_\beta(y)]|0)=-ie^{-\frac{1}{2}ip'(x+y)}S_{\mu\beta}{}^c(x-y);\quad(6.5\text{a})$$
$$(p'|\delta j_\mu(x)/\delta\psi_\beta(y)|0)=-e^{-\frac{1}{2}ip'(x+y)}S_{\mu\beta}{}^{(\text{adv})}(x-y);\quad(6.5\text{b})$$
$$(p'|\delta\bar{\eta}_\beta(y)/\delta A_\mu(x)|0)=-e^{-\frac{1}{2}ip'(x+y)}S_{\mu\beta}{}^{(\text{ret})}(x-y);\quad(6.5\text{c})$$
$$(p'|j_\mu(x)\bar{\eta}_\beta(y)|0)=-ie^{-\frac{1}{2}ip'(x+y)}S_{\mu\beta}{}^{(-)}(x-y);\quad(6.5\text{d})$$
$$(p'|\bar{\eta}_\beta(y)j_\mu(x)|0)=ie^{-\frac{1}{2}ip'(x+y)}S_{\mu\beta}{}^{(+)}(x-y).\quad(6.5\text{e})$$

The translation invariance of the field equations assures that the functions $S^{(i)}$ defined in this way are functions only of the difference $x-y$. Two useful relations among these functions are

$$S_{\mu\beta}{}^{(c)}(x)=S_{\mu\beta}{}^{(\text{adv})}(x)+S_{\mu\beta}{}^{(-)}(x)=S_{\mu\beta}{}^{(\text{ret})}(x)-S_{\mu\beta}{}^{(+)}(x),\quad(6.6\text{a})$$

and

$$S_{\mu\beta}{}^{(\text{ret})}(x)=S_{\mu\beta}{}^{(\text{adv})}(x)=S_{\mu\beta}{}^{(+)}(x)+S_{\mu\beta}{}^{(-)}(x).\quad(6.6\text{b})$$

In terms of the Fourier transform $G^{(i)}(k)$,

$$S^{(i)}(x)=\frac{1}{(2\pi)^4}\int e^{ikx}G^{(i)}(k)d^4k,$$
$$G^{(i)}(k)=\int e^{-ikx}S^{(i)}(x)d^4x,\quad(6.7)$$

the form-factor Eq. (6.2) is written

$$\bar{u}(p',s')F_\mu(p',q;p)u(p,s)=2\pi i\delta(p-q-p')G_{\mu\beta}{}^{(i)}(\tfrac{1}{2}p'+q)u(p,s).\quad(6.8)$$

The quantity of physical interest is this form factor considered as a function of positive q^2 when

$$p^2=p'^2=-M^2,\quad(6.9)$$

where M is the nucleon mass.

Because of momentum conservation (or translation invariance) at least one momentum in addition to q_μ must be varied. The representation Eq. (6.2) we have constructed is most convenient when p' is held fixed, because then the dependence on momentum transfer is contained entirely in the exponential factor

$$e^{-iqx}e^{ipy}=e^{-i(q+\frac{1}{2}p')(x-y)}e^{-\frac{1}{2}ip'(x+y)},\quad(6.10)$$

which has been used to obtain Eq. (6.8). We shall therefore use the rest system of the final nucleon with the following notation:

$$p_\mu'=(0,M),\quad(6.11\text{a})$$
$$q_\mu=(\lambda\mathbf{e},+\omega),\quad(6.11\text{b})$$
$$p_\mu=(-\lambda\mathbf{e},E=M+\omega).\quad(6.11\text{c})$$

The condition that p_μ be a nucleon momentum leaves ω the only variable (beside the trivial possibility of rotating $\mathbf{e}$), because λ is determined by Eqs. (6.9) and (6.11c) to be

$$\lambda=[(M+\omega)^2-M^2]^{\frac{1}{2}}.\quad(6.12)$$

7.—In order to establish a dispersion relation we should now like to apply Cauchy's theorem to $G_{\mu\beta}{}^{(c)}(\frac{1}{2}p'+q)$ considered as a function of complex ω. The Fourier integral Eq. (6.7),

$$G_{\mu\beta}{}^{(c)}(\tfrac{1}{2}p'+q)=\int e^{-\frac{1}{2}ip'x}\exp[-i(\lambda\mathbf{e}\cdot\mathbf{x}-\omega x_0)]\times S_{\mu\beta}{}^{(c)}(x)d^4x,\quad(7.1)$$

unfortunately exists only on part of the real axis,

$$\operatorname{Im}\omega=0,\quad\operatorname{Re}\omega>0\quad\text{or}\quad\operatorname{Re}\omega<-2M,\quad(7.2)$$

where

$$|\operatorname{Im}\omega|\geq|\operatorname{Im}\lambda|.\quad(7.3)$$

It is necessary to determine, therefore, whether there is an analytic function which is equal to the integral in Eq. (7.1) where that exists, and to locate its singularities if it can be found. In perturbation theory $S^{(c)}(x)$ and $G^{(c)}(k)$ can be exhibited explicitly as polynomials in the momentum components times simple functions of the invariant squares of the momenta, so that the analytic continuation can be carried out by inspection.[7]

We shall proceed by establishing a dispersion relation for nonphysical values of p^2,

$$p^2=-\tau,\quad\tau<0,\quad(7.4)$$

so that Eq. (6.12) becomes

$$\lambda=[(M+\omega)^2-\tau]^{\frac{1}{2}}. \quad (7.5)$$

We then conjecture that the analytic continuations of the $G^{(i)}$ as a function of τ can be extended to

$$\tau\leq M^2, \quad (7.6)$$

for which they have the required values given by Eq. (7.1) and its obvious modifications.

The functions $G^{(\pm)}$ are the easiest to discuss. By introducing into Eqs. (6.5d) and (6.5e) a complete set of states labeled by the quantum number n, with the rest-energy M_n, and the energy-momentum vector $k_\mu(k^2=-M_n^2)$ we obtain

$$S_{\mu\beta}^{(+)}(x)=-\frac{i}{(2\pi)^3}\sum_n\int d^4k\, 2k_0\theta(k_0)\delta(k^2+M_n^2)$$
$$\times e^{ix(\frac{1}{2}p'-k)}\langle p'|\bar{\eta}_\beta(0)|n,k\rangle\langle n,k|j_\mu(0)|0\rangle, \quad (7.7a)$$

and

$$S_{\mu\beta}^{(-)}(x)=\frac{i}{(2\pi)^3}\sum_n\int d^4k\, 2k_0\theta(k_0)\delta(k^2+M_n^2)$$
$$\times e^{-ix(\frac{1}{2}p'-k)}\langle p'|j_\mu(0)|nk\rangle\langle nk|\bar{\eta}_\beta(0)|0\rangle. \quad (7.7b)$$

In view of Eq. (7.4), q^2 and the momentum λ are now given by

$$q^2=M^2+2M\omega-\tau,\quad \lambda=[(M+\omega)^2-\tau]^{\frac{1}{2}}, \quad (7.8)$$

whence follows

$$|\mathrm{Im}\,\omega|\geq|\mathrm{Im}\,\lambda|, \quad (7.9)$$

for all values of ω, real and complex. The Fourier transforms at the value $\frac{1}{2}p'+q$ can therefore be obtained as

$$G_{\mu\beta}^{(+)}(\tfrac{1}{2}p'+q)\equiv G_{\mu\beta}^{(+)}(\omega,\tau)$$
$$=2\pi i\sum_{n+}2\omega\theta(-\omega)\delta(q^2+M_{n+}^2)$$
$$\times\langle p'|\bar{\eta}_\beta(0)|n+,-q\rangle\langle n+,-q|j_\mu(0)|0\rangle \quad (7.10)$$

and

$$G_{\mu\beta}^{(-)}(\tfrac{1}{2}p'+q)=G_{\mu\beta}^{(-)}(\omega,\tau)$$
$$=2\pi i\sum_{n-}2E\theta(E)\delta(p^2+M_{n-}^2)$$
$$\times\langle p'|j_\mu(0)|n-,p\rangle\langle n-,p|\bar{\eta}_\beta(0)|0\rangle, \quad (7.11)$$

where we have introduced notation to indicate the functional dependence on ω,τ and to distinguish the sets of states $n\pm$ that contribute to $G^{(\pm)}$ respectively.

Since $j_\mu(0)$ is a vector operator, its matrix element $\langle n+,q|j_\mu(0)|0\rangle$ vanishes unless the state $n+$ contains at least two pseudoscalar mesons; hence the lowest-energy intermediate state has $M_{n+}=2m_\pi$ and the function $G^{(+)}(\omega,\tau)$ vanishes over a large part of the real axis,

$$G_{\mu\beta}^{+}(\omega,\tau)=0,\quad q^2>-4m_\pi^2$$
$$\text{or }\omega>\omega_0(\tau)=\frac{-M^2+\tau-4m_\pi^2}{2M}. \quad (7.12)$$

In particular, $G^{(+)}$ vanishes in the "physical" region $\omega\geq0$ where the quantum is really emitted ($E\geq M$). Similarly, the matrix element $\langle n-,p|\bar{\eta}_\beta(0)|0\rangle$ vanishes unless the state $n-$ contains at least one nucleon and a meson, $M_n\geq(M+m_\pi)$, so that we have

$$G_{\mu\beta}^{(-)}(\omega,\tau)=0,\quad p^2\geq-(M+m_\pi)^2$$
$$\text{or }\tau<(M+m_\pi)^2. \quad (7.13)$$

Since the inequality is always satisfied in the region (7.6) in which we are interested, the function $G^{(-)}$ will not be considered further.

From Eq. (6.6) we may now infer the corresponding relations among the Fourier transforms,

$$G_{\mu\beta}^{(c)}(\omega,\tau)=G_{\mu\beta}^{(\mathrm{ret})}(\omega,\tau)$$
$$=G_{\mu\beta}^{(\mathrm{adv})}(\omega,\tau),\quad \omega>\omega_0(\tau), \quad (7.14)$$

and

$$G_{\mu\beta}^{(\mathrm{ret})}(\omega,\tau)-G_{\mu\beta}^{(\mathrm{adv})}(\omega,\tau)$$
$$=G_{\mu\beta}^{(+)}(\omega,\tau),\quad \omega<\omega_0(\tau). \quad (7.15)$$

8.—To construct the analytic continuation of the retarded and advanced functions we must exploit their space-time behavior, Eq. (6.4). To avoid the branch points at $\lambda=0$ in the integral representations Eq. (7.1), we shall treat the even and odd forms, i.e., the forms symmetrized and antisymmetrized in the sense of the vector **e**,

$${}^{(e,o)}G_{\mu\beta}^{(\mathrm{ret})}(\omega,\tau)=\int e^{-\frac{1}{2}ip'x}e^{i\omega x}$$
$$\times\left(\cos\lambda\mathbf{e}\cdot\mathbf{x},\frac{-i}{\lambda}\sin\lambda\mathbf{e}\cdot\mathbf{x}\right)S_{\mu\beta}^{(\mathrm{ret})}(x)d^4x, \quad (8.1)$$

which are even functions of λ. Because $S^{(\mathrm{ret})}(x)$ restricts the integration to the future light cone, these functions are analytic in the region

$$\mathrm{Im}\,\omega>|\mathrm{Im}\,\lambda|, \quad (8.2a)$$

or

$$\mathrm{Im}\,\omega>0, \quad (8.2b)$$

in view of Eq. (7.9). The functions

$${}^{(e,o)}G_{\mu\beta}^{(\mathrm{adv})}(\omega,\tau)=\int e^{-\frac{1}{2}ip'x}e^{i\omega x_0}$$
$$\times\left(\cos\lambda\mathbf{e}\cdot\mathbf{x},\frac{-i}{\lambda}\sin\lambda\mathbf{e}\cdot\mathbf{x}\right)S_{\mu\beta}^{(\mathrm{adv})}(x)d^4x \quad (8.3)$$

are analytic in the region

$$\mathrm{Im}\,\omega<-|\mathrm{Im}\,\lambda| \quad (8.4a)$$

or

$$\operatorname{Im}\omega<0 \tag{8.4b}$$

because the integration extends only over the past light cone.

Furthermore, Eq. (7.14) states that on part of the real axis

$$\omega>\omega_0(\tau) \tag{8.5}$$

the causal, advanced, and retarded functions are equal; they are, therefore, the same analytic function ${}^{(e,o)}\tilde{G}_{\mu\beta}(\omega,\tau)$, regular in the entire complex ω plane with a branch point at

$$\omega=\omega_0(\tau), \tag{8.6}$$

and a cut from there to infinity; we shall take the cut along the negative real axis. A retarded function is obtained by approaching the cut from the upper half plane and an advanced or causal function by approaching from the lower half plane.

Since the discontinuity in $\tilde{G}$ on the cut is known from Eq. (7.15), we can apply the Cauchy theorem to the function ${}^{(e)}\tilde{G}(\omega)/\omega$ if this function approaches zero for large values of ω. In accordance with the discussion in Sec. I, 2, we assume that this is the case, and that ${}^{(e)}\tilde{G}(\omega)$ may not approach zero. The resulting dispersion relation or spectral representation is

$${}^{(e)}\tilde{G}_{\mu\beta}(\omega,\tau)=\frac{\omega}{2\pi i}\int_{-\infty}^{\omega_0(\tau)}\frac{G_{\mu\beta}{}^{(+)}(\omega',\tau)}{\omega'(\omega'-\omega)}d\omega'+{}^{(e)}\tilde{G}_{\mu\beta}(0,\tau),\quad \omega>0. \tag{8.7}$$

It follows from covariance arguments that the function ${}^{(o)}\tilde{G}$ approaches zero for large values of ω if ${}^{(e)}\tilde{G}(\omega)/\omega$ does. The spectral representation for that function is therefore

$${}^{(o)}G_{\mu\beta}(\omega,\tau)=\frac{1}{2\pi i}\int_{-\infty}^{\omega_0(\tau)}\frac{{}^{(o)}G_{\mu\beta}{}^{(+)}(\omega,\tau)}{\omega'-\omega}d\omega',\quad \omega>0. \tag{8.8}$$

9.—We shall now assume that Eqs. (8.7) and (8.8) hold for

$$\tau=M^2. \tag{9.1}$$

The possibility of such an analytic continuation has been proved rigorously[12] only for the case in which the meson and nucleon masses satisfy the inequality

$$m_\pi>(\sqrt{2}-1)M.$$

We believe that this restriction is a result of the method used for the proof and that it will be removed when further progress is made in the study of this problem.

We may therefore write the spectral representations for the even and odd form factors ${}^{(e,o)}F$, which differ from the functions ${}^{(e,o)}\tilde{G}$ only by the constant final nucleon spinor. It is still more useful first to decompose the form factor into the four real scalar functions described in the introduction,

$$F_\mu(p',q;p)=i\gamma_\mu[G_1{}^S(q^2)+\tau_3G_1{}^V(q^2)]+\sigma_{\mu\nu}q_\nu[G_2{}^S(q^2)+\tau_3G_2{}^V(q^2)], \tag{9.2}$$

and to do the same for $G^{(+)}$,

$$G_{\mu\beta}{}^{(+)}(\omega,M^2)=-2i\bar{u}_\alpha(p',s)\{i\gamma u[g_1{}^S(-q^2)+\tau_3g_1{}^V(-q^2)]+\sigma_{\mu\nu}q_\nu[g_2{}^S(-q^2)+\tau_3g_2{}^V(-q^2)]\}. \tag{9.3}$$

It is clear that each of the functions $g_{1,2}{}^{S,V}$ is related to the analytic continuation of the corresponding $G_{1,2}{}^{S,V}$ in the same way as $G^{(+)}$ is related to $G^{(c)}$. Then the $g_{1,2}{}^{S,V}$ are real functions as a consequence of the Riemann-Schwarz "principle of reflection."

Each of the two functions $G_1{}^{S,V}$ has a spectral representation of the form Eq. (8.7), in which we may set [Eq. (7.8)]

$$\omega=q^2/2M,\quad \omega'=-m^2/2M, \tag{9.4}$$

to obtain Eqs. (2.3) and (2.4). The functions $q_iG_2{}^{S,V}$ satisfy a representation of the form (8.8), while $q_0G_2{}^{S,V}$ satisfy one of the form (8.7) with $[q_0G_2{}^{S,V}]_{q_0=0}=0$; both give the same result, which leads to Eq. (2.5) and (2.6) after the change of variables in Eq. (9.4).

III. GENERAL PROPERTIES OF INTERMEDIATE-STATE CONTRIBUTIONS

10.—Having obtained the spectral representation for the form factors, we now focus our attention on the four weight functions

$$g_{1,2}{}^{S,V}(m^2). \tag{10.1}$$

As already observed, these functions contain a sum of terms corresponding to the possible intermediate states in Formula (7.10); thus for each g,

$$g=g_{(2\pi)}+g_{(3\pi)}+\cdots+g_{(2K)}+\cdots+g_{(N\bar{N})}+\cdots. \tag{10.2}$$

Each of these partial weight functions g_i vanishes for value of m^2 less than $(m_i)^2$, where m_i is the sum of the rest masses of the particles in the state i. As argued in the introduction, a particular g_i therefore contributes to the nucleon structure only within radii of the order of the Compton wavelength associated with the mass m_i. Thus it is appropriate to concentrate on the functions g_i corresponding to the low-mass intermediate states, in order to discuss the outer regions of the nucleon in configuration space.

According to (7.10) the weight functions g_i are proportional to the matrix element for a virtual photon of mass m to "decay" into the intermediate state in question. It follows that the total angular momentum of any possible intermediate state is one, while under either space inversion or charge conjugation the state must be odd. The total charge is of course zero. Furthermore the total isotopic spin I can be only zero or one, states with $I=0$ contributing to the isotopic scalar part of the nucleon electromagnetic structure and those with $I=1$ to the isotopic vector part.

To obtain a special selection rule for the least massive states, those containing only pions, one may consider the combined operation of charge conjugation and 180° rotation about the y axis in isotopic spin space. Under this operation a pion state is even or odd depending only on whether an even or an odd number of pions is present. Since the states of interest here are odd under charge conjugation, they will contain an even number of pions if the total isotopic spin is odd and an odd number of pions if I is even, i.e.,

$$g_{(n\pi)}{}^{S}=0, \quad \text{if } n \text{ is even}, \tag{10.3}$$

and

$$g_{(n\pi)}{}^{V}=0, \quad \text{if } n \text{ is odd}. \tag{10.4}$$

We thus have

$$\begin{aligned} g^{S}&=g_{(3\pi)}+g_{(5\pi)}+\cdots+g_{(2K)}{}^{S}+\cdots+g_{(N\bar{N})}{}^{S}+\cdots,\\ g^{V}&=g_{(2\pi)}+g_{(4\pi)}+\cdots+g_{(2K)}{}^{V}+\cdots\\ &\qquad +g_{(N\bar{N})}{}^{V}+\cdots. \end{aligned} \tag{10.5}$$

If the charge contribution from the three-pion intermediate state were of the same magnitude as that of the two-pion state, one might have an explanation for the difference in second radial moments between the proton and neutron charge distributions. There is no visible reason, of course, why the three-pion configuration should contribute substantially to the charge density and not at the same time to the magnetic moment. However, this same statement can be made in our current state of knowledge about any possible source of isotopic scalar charge, so that the three-pion state must be regarded as a possible candidate to supply the needed scalar charge. At the present time we know of no sensible way to estimate even the sign of the three-pion contribution. On one side a closed nucleon loop is required to couple this system to the electromagnetic field and on the other side a nonphysical matrix element for the process $3\pi\rightarrow N+\bar{N}$ (or $\pi+N \rightarrow 2\pi+N$) is involved.

For intermediate states of mass greater than $2M$ the other factor in g_i, as given by (7.10), is the physical transition amplitude connecting the state i to a nucleon-antinucleon pair. According to calculations by Bernstein, Federbush, Goldberger, and Treiman,[13] the unitarity of the S matrix severely limits the size of contributions from such states, a circumstance that gives encouragement to a program of calculation which ignores the high mass region. In particular, the contribution from the nucleon-antinucleon intermediate state is given by the product of nucleon electromagnetic structure factors themselves, evaluated at $q^2=-m^2$, and nucleon-antinucleon elastic scattering amplitudes in the physical region. There seems no reason to think that this $(N\bar{N})$ contribution should be even remotely approximated by setting $G_2{}^{S,V}(-m^2)=0, G_1{}^{S,V}(-m^2)=\frac{1}{2}e$, and using the Born approximation for nucleon-antinucleon scattering, the procedure equivalent to the standard perturbation calculations.[1] This point is discussed further in Sec. VI.

[13] J. Bernstein and M. L. Goldberger, paper delivered at the 1957 Stanford Conference on Nuclear Sizes (unpublished).

IV. FORMULATION OF THE TWO-PION CONTRIBUTION

11.—The contribution to the nucleon electromagnetic structure from the two-meson intermediate state will be calculated in the next two sections. The reasons for concentrating on this part of the process are: (1) it is the only part for which at present anything like a calculation is feasible; (2) there is reason to hope, as explained in the introduction, that the two-meson contribution dominates the magnetic moment.

For an intermediate state consisting of mesons of four-momenta q_1 and q_2 with isotopic spin indices j and k, we have for the spectral distribution function, Eq. (7.10),

$$\begin{aligned} &\bar{u}(p')I_\mu{}^{(2\pi)}(p',p)u(p)\\ &=-\frac{1}{2i}[G_{\mu\beta}{}^{(+)}(\omega,M^2)\mu_\beta(p)]^{(2\pi)}\\ &=-\tfrac{1}{2}\pi\sum_{jk}\frac{1}{(2\pi)^3}\int d^3q_1d^3q_2\delta(q_1+q_2+q)\\ &\qquad\times\langle p|\bar{\eta}(0)|q_1jq_2k\rangle u(p)\langle q_1j,\, q_2k|j_\mu(0)|0\rangle. \end{aligned} \tag{11.1}$$

The second factor of the integrand, i.e., the matrix element describing the disappearance of the photon with the creation of a pion pair, may on the basis of invariance considerations be written

$$\begin{aligned} \langle q_1j,\, q_2k|j_\mu(0)|0\rangle&=\frac{-ie}{(4\omega_1\omega_2)^{\frac{1}{2}}}(q_1-q_2)_\mu\\ &\qquad\times(\delta_{j1}\delta_{k2}-\delta_{j2}\delta_{k1})F_\pi[(q_1+q_2)^2]. \end{aligned} \tag{11.2}$$

Here $F_\pi[(q_1+q_2)^2]$ is a form factor associated with the one-photon, two-meson vertex, normalized to unity for zero argument. This function for positive argument describes the electromagnetic structure of the pion in the same sense as the functions $G_{1,2}{}^{S,V}$ describe the nucleon, and in principle could be measured directly by electron-pion elastic scattering. In practice we shall be forced to set F_π equal to unity ("point pion" approximation) since there is at present no understanding, either experimental or theoretical, of the pion structure. However, this approximation may be postponed until the very end of the calculation, since $F_\pi(-m^2)$ appears simply as a multiplicative factor in the weight functions $g_{2\pi}(m^2)$.

The other factor in the integrand of Eq. (11.1) is related to the meson-nucleon scattering amplitude. Explicitly, if the amplitude for scattering a meson in the state q by a nucleon in the state p, leading to a meson

q' and a nucleon p', is denoted by $\langle p',q'|T|p,q\rangle$, then we have

$$\langle p'|\bar{\eta}(0)|q_1j,\, q_2k\rangle u(p)=\langle p_1',-q_2k|T|p,\, q_1j\rangle. \quad (11.3)$$

Of course, writing $-q_2$ for the final pion energy-momentum implies an extension of the scattering amplitude to a nonphysical region. This extension will occupy Sec. III, 14.

Using the notation introduced by Chew, Goldberger, Low, and Nambu,[14]

$$\begin{aligned}\langle p',\, -q_2k|T|p,\, q_1j\rangle &\\ =\frac{1}{(4\omega_1\omega_2)^{\frac{1}{2}}}&(\bar{u}(p')\{[-A^{(+)}(W^2,q^2)\\ &+i\gamma_\mu Q_\mu B^{(+)}(W^2,q^2)]\delta_{jk}+[-A^{(-)}(W^2,q^2)\\ &+i\gamma_\mu Q_\mu B^{(-)}(W^2,q^2)]\tfrac{1}{2}[\tau_k,\tau_j]\}u(p)), \quad (11.4)\end{aligned}$$

where $-q=(q_1+q_2)$, $Q=\frac{1}{2}(q_1-q_2)$, $P=\frac{1}{2}(p+p')$, and $W^2=-(P+Q)^2$, we may carry out the isotopic-spin sum in Eq. (11.1) to obtain

$$\begin{aligned}I_\mu^{(2\pi)}(p',p)=-\frac{e}{4\pi^2}\tau_3\int d^4q_1d^4q_2\delta(q_1+q_2+q)&\\ \times\delta(q_1^2+m_\pi^2)\delta(q_2^2+m_\pi^2)\tfrac{1}{2}(q_1-q_2)_\mu&\\ \times[A^{(-)}(W^2,q^2)-i\gamma_\lambda Q_\lambda B^{(-)}(W^2,q^2)]F_\pi(q^2). &\quad (11.5)\end{aligned}$$

The three-dimensional integrals over q_1 and q_2 have been increased to four dimensions by adding the mass-shell delta functions. It can be seen that only positive frequencies contribute.

12.—It should be noted that in Eq. (11.5) only the charge-exchange scattering amplitudes occurs and that the contribution, as expected, is only to the isotopic vector part of the nucleon electromagnetic structure. Introducing q and Q in place of q_1 and q_2 and then performing the integration over d^4q, we obtain

$$\begin{aligned}I_\mu^{(2\pi)}(p',p)=-\frac{e}{4\pi^2}\tau_3\int d^4Q\,\delta[(\tfrac{1}{2}q+Q)^2+m_\pi^2]&\\ \times\delta[(\tfrac{1}{2}q-Q)^2+m_\pi^2]Q_\mu[A^{(-)}(W^2,q^2)&\\ -i\gamma_\lambda Q_\lambda B^{(-)}(W^2,q^2)]F_\pi(q^2). &\quad (12.1)\end{aligned}$$

The next task is to relate formula (12.1) to the scalar weight functions $g_{1,2(2\pi)}{}^V(m^2)$. Clearly we are to make the identification $m^2=-q^2$, and by standard invariance arguments we find

$$\begin{aligned}&g_{1(2\pi)}{}^V(m^2)\\ &=[M\alpha(m^2)+\beta_1(m^2)+M^2\beta_2(m^2)]\frac{e}{4\pi^2}F_\pi(-m^2), \quad (12.2)\\ &g_{2(2\pi)}{}^V(m^2)\\ &=[-\tfrac{1}{2}\alpha(m^2)-\tfrac{1}{2}M\beta_2(m^2)]\frac{e}{4\pi^2}F_\pi(-m^2), \quad (12.3)\end{aligned}$$

where

$$\begin{aligned}\alpha(-q^2)=-\int d^4Q\delta(Q^2+\tfrac{1}{4}q^2+m_\pi^2)\delta(2q_\lambda Q_\lambda)&\\ \times(P_\nu Q_\nu/P^2)A^{(-)}(W^2,q^2), &\quad (12.4)\\ \beta_1(-q^2)=\int d^4Q\delta(Q^2+\tfrac{1}{4}q^2+m_\pi^2)\delta(2q_\lambda Q_\lambda)&\\ \times\{[P^2Q^2-(PQ)^2]/2P^2\}B^{(-)}(W^2,q^2), &\quad (12.5)\\ \beta_2(-q^2)=\int d^4Q\delta(Q^2+\tfrac{1}{4}q^2+m_\pi^2)\delta(2qQ)&\\ \times\{[P^2Q^2-3(PQ)^2]/2(P^2)^2\}B^{(-)}(W^2,q^2). &\quad (12.6)\end{aligned}$$

If we recall that

$$P^2=-M^2-\tfrac{1}{4}q^2, \quad (12.7)$$

then it is clear that these integrals indeed depend only on the single scalar q^2.

13.—In order to exploit these results, the pion-nucleon scattering amplitude is needed in a nonphysical region, in particular in a region where the square of the momentum transfer q^2 is negative. The variable $W^2=-(P+Q)^2$ also takes on nonphysical values, but the dispersion relations permit us to extend the scattering amplitude to values of W^2 anywhere in the complex plane. It is the technique of extension to negative q^2 that is our particular problem here.

Actually, for q^2 less than $-4M^2$, the matrix element we are concerned with can be identified with the physical amplitude for the process $\pi+\pi\rightarrow N+\bar{N}$. In the future this identification may turn out to be useful, but at the moment the only good theoretical approach we have to pion-nucleon matrix elements derives from the prominence of the $(\frac{3}{2},\frac{3}{2})$-state scattering resonance. Any calculation attempted now has to be based on pion-nucleon scattering rather than on pion-pion production of a nucleon pair. This conclusion is reinforced by the empirical fact, emphasized in the introduction, that the "average" value of m^2 in the magnetic-moment weight function $g_2{}^V(m^2)$ is less than M^2, so that we may hope not to have to be concerned with values of $(-q^2)=m^2$ that are greater than $4M^2$. If the high-virtual-mass region turns out to be crucial in understanding the nucleon magnetic moment, our motivation for concentrating on the two-pion contribution will be lost.

So long as one works with Feynman diagrams, i.e., with a perturbation evaluation of the pion-nucleon matrix element, there is no problem about continuing to negative values of q^2; the functional dependence is explicit. The whole point of the approach adopted here, however, is to avoid the perturbation method; and the most obvious alternative is the method already used with some success in the dispersion relations for non-forward scattering, where nonphysical values of q^2 also occur—that is, an extension by means of Legendre polynomials.

It is clear that a polynomial expansion cannot be

[14] Chew, Goldberger, Low, and Nambu, Phys. Rev. **106**, 1337 (1957).

valid for indefinitely high values of $|q^2|$. In fact, as pointed out by Symanzik, the possibility of meson-meson scattering implies that strictly speaking the expansion will not converge for $|q^2|>4m_\pi^2$, a condition which excludes our entire range of integration. However, there is reason to think that meson-meson scattering is weak and that for practical purposes the Legendre expansion may be used for $|q^2|<4M^2$. We shall make this optimistic assumption here, thereby allowing a crude calculation of $g_{2\pi}(m^2)$ in the low-mass region.

14.—Let us start with the conventional scattering dispersion relations in the form used by Chew, Goldberger, Low, and Nambu[14]:

$$A^{(-)}(W^2,q^2)=\frac{1}{\pi}\int_{(M+m_\pi)^2}^{\infty} dW'^2\,\mathrm{Im}\,A^{(-)}(W'^2,q^2)\times\left\{\frac{1}{W'^2+(P+Q)^2}-\frac{1}{W'^2+(P-Q)^2}\right\}. \quad (14.1)$$

$$B^{(-)}(W^2,q^2)=g_r^2\left[\frac{1}{M^2+(P+Q)^2}+\frac{1}{M^2+(P-Q)^2}\right]+\frac{1}{\pi}\int_{(M+m_\pi)^2}^{\infty} dW'^2\,\mathrm{Im}\,B^{(-)}(W'^2,q^2)\times\left\{\frac{1}{W'^2+(P+Q)^2}+\frac{1}{W'^2+(P-Q)^2}\right\}, \quad (14.2)$$

where g_r^2 is the rationalized and renormalized Yukawa coupling constant. As emphasized by these authors, the forms (14.1) and (14.2) correspond to the most optimistic assumption possible about the behavior of the amplitudes as W^2 approaches infinity. There is, however, some experimental evidence to support the optimistic assumption for charge-exchange scattering, the case with which we are dealing here.[15] We note for future reference that the Born approximation; i.e., neglect of $A^{(-)}$ and of the integral contribution to $B^{(-)}$ in the calculation of the weight functions Eqs. (12.2) to (12.6), is equivalent to including in the nucleon form factor only the lowest-order perturbation-theory contribution to the meson current effects.

The only place in Eqs. (14.1) and (14.2) where the dependence on q^2 is not explicit is in the imaginary parts of $A^{(-)}$ and $B^{(-)}$ in the dispersion integrals. It is here that the polynomial expansion is needed. According to Chew *et al.*,[14] these imaginary parts may be expressed in terms of partial-wave "total" charge-exchange cross sections $\sigma_{l\pm}^{(-)}$ for states with parity $(-1)^{l+1}$ and total angular momentum $l\pm\frac{1}{2}$,

$$\frac{1}{k}\,\mathrm{Im}\,(A^{(-)};B^{(-)})=\frac{(W+M;1)}{E+M}\sum_{l=0}^{\infty}\left[P_{l+1}'(x)\sigma_{l+}^{(-)}-P_{l-1}'(x)\sigma_{l-}^{(-)}\right]-\frac{(W-M;-1)}{E-M}\sum_{l=0}^{\infty}P_l'(x)\left[\sigma_{l-}^{(-)}-\sigma_{l+}^{(-)}\right]. \quad (14.3)$$

Here E is the nucleon energy and k the relative pion-nucleon momentum in the barycentric system, while $P_l'(x)$ are derivatives of Legendre polynomials of the cosine x of the scattering angle,

$$x=1-q^2/2k^2, \quad (14.4)$$

which exhibit the dependence on q^2.

Our intention, of course, is to use experimental information about the total cross sections for the first few partial waves in pion-nucleon scattering to effect an approximate evaluation of Eq. (14.3) and thus of Eqs. (15.1) and (14.2). Any single partial-wave contribution can then be extended to negative values of q^2. The difficulty, as explained above, is that the series in l does not converge for large $|q^2|$.

15.—In evaluating the quantities $\alpha(m^2)$, $\beta_1(m^2)$, and $\beta_2(m^3)$ as given by formulas (12.4) to (12.6), we may use the representations (14.1) and (14.2) to carry out the integration over Q, because they give the dependence on W and thus on Q. One obtains inverse trigonometric functions of a variable y,

$$y(W'^2,m^2)=\frac{2q_\pi q_n}{W'^2+q_\pi^2-q_n^2}, \quad (15.1)$$

where

$$q_\pi=(\tfrac{1}{4}m^2-m_\pi^2)^{\frac{1}{2}}, \quad q_n=(M^2-\tfrac{1}{4}m^2)^{\frac{1}{2}}. \quad (15.2)$$

The functions are

$$I_\alpha(W'^2,+m^2)=\frac{\pi}{m}\left(\frac{q_\pi}{q_n^2}\right)\left[1-\frac{1}{y}\tan^{-1}y\right], \quad (15.3)$$

$$I_{\beta 1}(W'^2,+m^2)=\frac{\pi}{m}\frac{q_\pi^2}{2q_n}\left[\tan^{-1}y-\frac{1}{y}\left(1-\frac{1}{y}\tan^{-1}y\right)\right], \quad (15.4)$$

$$I_{\beta 2}(W'^2,+m^2)=\frac{-\pi}{m}\frac{q_\pi^2}{2q_n^3}\left[\tan^{-1}y-\frac{3}{y}\left(1-\frac{1}{y}\tan^{-1}y\right)\right], \quad (15.5)$$

[15] Goldberger, Miyazawa, and Oehme, Phys. Rev. **99**, 986 (1955).

and the final formulas for the spectral functions are

$$g_{1}{}^{V}{}_{(2\pi)}(m^2)=\frac{e}{4\pi^2}F_\pi(-m^2)\bigg\{g_r{}^2[I_{\beta1}(M^2,+m^2)+M^2I_{\beta2}(M^2,+m^2)]+\frac{1}{\pi}\int_{(M+m_\pi)^2}^{\infty}dW'^2[M\,\mathrm{Im}\,A^{(-)}(W'^2,-m^2)\times I_\alpha(W'^2,+m^2)+\mathrm{Im}\,B^{(-)}(W'^2,-m^2)(I_{\beta1}(W'^2,+m^2)+M^2I_{\beta2}(W'^2,+m^2))]\bigg\}, \quad (15.6)$$

$$g_{2}{}^{V}{}_{(2\pi)}(m^2)=\frac{e}{4\pi^2}F_\pi(-m^2)\bigg\{-\frac{M}{2}g_r{}^2I_{\beta2}(M^2,+m^2)+\frac{1}{\pi}\int_{(M+m_\pi)^2}^{\infty}dW'^2[-\tfrac{1}{2}\,\mathrm{Im}\,A^{(-)}(W'^2,-m^2)I_\alpha(W'^2,+m^2)-\tfrac{1}{2}M\,\mathrm{Im}\,B^{(-)}(W'^2,-m^2)I_{\beta2}(W'^2,+m^2)]\bigg\}. \quad (15.7)$$

This is as far as one can carry the calculation without making approximations.

V. ATTEMPTED EVALUATION OF THE TWO-PION CONTRIBUTION

16.—Several different kinds of approximations may be distinguished. First one may take advantage of the dominance of the $(\frac{3}{2},\frac{3}{2})$ resonance in Eqs. (15.6) and (15.7) to treat $W'-M$ as small compared with M, this being the approach which has had considerable success in theoretical discussions of pion-nucleon scattering[14] and photopion production.[3] Of course for making practical use of the polynomial expansions it is also necessary that $m^2=-q^2$ be small—an unfortunate requirement, since for calculation of the electromagnetic form factors an integral over all values of m^2 is involved. Nevertheless, it may be of interest to see how the weight functions $g_{(2\pi)}(m^2)$ behave for small m^2; therefore we tentatively neglect all terms of order $1/M^2$ and assume $|q^2|$ sufficiently small that the polynomial expansions are well approximated by keeping only S and P waves. Formula (14.3) then becomes

$$\mathrm{Im}\,A^{(-)}(W'^2,q^2)\approx k'\left[\sigma_S{}^{(-)}+3\sigma_{P\frac{3}{2}}{}^{(-)}\left(1-\frac{q^2}{2k'^2}\right)\right]-\frac{2M\omega'}{k'}[\sigma_{P\frac{1}{2}}{}^{(-)}-\sigma_{P\frac{3}{2}}{}^{(-)}], \quad (16.1)$$

$$\mathrm{Im}\,B^{(-)}(W'^2,q^2)\approx\frac{2M}{k'}[\sigma_{P\frac{1}{2}}{}^{(-)}-\sigma_{P\frac{3}{2}}{}^{(-)}], \quad (16.2)$$

where $\omega'=W'-M$.

Furthermore, in view of the uncertainties involved it seems legitimate to set all partial cross sections for pion-nucleon scattering, except that for the $(\frac{3}{2},\frac{3}{2})$ state, equal to zero and to approximate the latter by a delta function. From the effective-range approach[16] one may relate integrals over the $(\frac{3}{2},\frac{3}{2})$ resonance to the value of the coupling constant $g_r{}^2$, *viz.*,

$$\frac{1}{\pi}\frac{2M}{k'}\sigma_{P\frac{3}{2}}{}^{(-)}\approx-\frac{4}{9}g_r{}^2\delta[W'^2-(M+\omega_r)^2], \quad (16.3)$$

[16] G. Chew and F. Low, Phys. Rev. **101**, 1570 (1956).

where ω_r is the resonance energy $(\omega_r\approx 2m_\pi)$. In this way we approximately evaluate the integrals over dW'^2 in Eqs. (15.6) and (15.7) to find

$$g_{1(2\pi)}{}^{V}(m^2)\approx\frac{ef^2}{m_\pi{}^2}\left(\frac{2q_\pi}{m}\right)\left\{\left(\frac{m^2}{2}-m_\pi{}^2\right)-\frac{8}{9}\left(\omega_r{}^2+\frac{m^2}{2}-m_\pi{}^2\right)\left[1-\frac{\omega_r}{q_\pi}\tan^{-1}\left(\frac{q_\pi}{\omega_r}\right)\right]\right\}, \quad (16.4)$$

$$g_{2(2\pi)}{}^{V}(m^2)\approx\frac{ef^2}{m_\pi{}^2}\left(\frac{q_\pi{}^2}{m}\right)\times\left\{\frac{\pi}{2}+\frac{4}{9}\left[\frac{\omega_r{}^2+q_\pi{}^2}{q_\pi{}^2}\tan^{-1}\left(\frac{q_\pi}{\omega_r}\right)-\frac{\omega_r}{q_\pi}\right]\right\}, \quad (16.5)$$

where

$$f^2=\left(\frac{m_\pi}{2M}\right)^2\left(\frac{g_r{}^2}{4\pi}\right)\approx 0.08.$$

In Eqs. (16.4) and (16.5) only terms of lowest order in $1/M$ have been kept, in order to achieve simple formulas and to facilitate comparison with the cut off model. The first terms in the large brackets are due to the nucleon poles and are seen to be substantially larger than the contribution from the $(\frac{3}{2},\frac{3}{2})$ resonance.

The general form of these approximate (no-recoil, low-m^2) expressions is similar to results obtained by several authors using the cutoff model.[5,17] In that model one finds for the electromagnetic structure factor the lowest-order perturbation result plus relatively small corrections proportional to integrals over charge-exchange scattering cross sections. For the magnetic moment it is the spin-flip cross section that occurs, while for the charge it is the non-spin-flip, the same forms obtained here. If our expressions (16.4) and (16.5) are cut off at $m^2\sim(2M)^2$, numerical results close to those of the cutoff model emerge.[18]

[17] S. Treiman and R. Sachs, Phys. Rev. **103**, 435 (1956); G. Salzman, Phys. Rev. **105**, 1076 (1957).

[18] The algebraic forms of the structure factors so far derived from the cutoff model are more complicated than the spectral representations (2.4) and (2.6) even though the numerical content is approximately equivalent when the approximations (16.4) and (16.5) are employed.

Because of the approximations made, the results (16.4) and (16.5) have incorrect asymptotic behavior. Instead of vanishing at infinity, $g_{1(2\pi)}(m^2)/m^2$ approaches a constant while $g_{2(2\pi)}(m^2)$ increases as m. As explained already, it is not easy to remedy this defect because the polynomial expressions (14.3) and (14.4) are inappropriate for asymptotic considerations, and so long as the behavior at infinity is wrong we cannot calculate the electromagnetic structure factors without cutting off. For pion-nucleon scattering[14] and photopion production[3] it was possible, by use of the spectral-representation approach to local-field theory, to reproduce the essential results of the cutoff model once the position of the $(\frac{3}{2},\frac{3}{2})$ resonance was known. It was not necessary to introduce a cutoff explicitly. We have not been able to do the same here, and we infer that the cutoff model is correspondingly less reliable for describing the nucleon electromagnetic structure than it is for phenomena involving real pions of low energy.

17.—It is interesting to note, however, that in the low-m^2 region our results (16.4) and (16.5) are fairly well represented by making the Born approximation to the scattering amplitude, i.e., keeping only the rational term in Eq. (14.2) which comes from the single-nucleon intermediate state. For example, at the empirically determined "average" m^2 [see Eq. (3.7)] the contribution to the magnetic moment arising from the integral over the $(\frac{3}{2},\frac{3}{2})$ resonance is only 17%, according to Eq. (16.5). Once we recognize this simplifying fact, it is easily possible to evaluate the magnetic-moment form factor with no further approximations other than treating the π meson as a point charge. As stated earlier, the result is precisely equivalent to lowest-order perturbation theory.

It may seem remarkable that a perturbation result can be anywhere near the truth, since it is well known that the perturbation calculation of pion-nucleon scattering is grossly misleading. The main trouble for scattering, however, occurs for the non-charge-exchange amplitude, where the S-wave part is overestimated by an order of magnitude. The g^2 approximation to the charge-exchange amplitude, on the other hand, is not too bad at low energies even in the physical region and, in the nonphysical region required here, is relatively more accurate because one is closer to the pole at $W=M$ than to the $(\frac{3}{2},\frac{3}{2})$ resonance. In the immediate neighborhood of the pole, of course, the perturbation result is exact. The weight functions we obtain now without the neglect of nucleon recoil are

$$g_1{}^V(2\pi)\approx\frac{ef^2}{m_\pi{}^2}\left(\frac{2M^2q_\pi{}^2}{mq_n}\right)\left\{\frac{2}{y_0}\left(1-\frac{1}{y_0}\tan^{-1}y_0\right)\right.$$
$$\left.-\frac{m^2}{4q_n{}^2}\left[\tan^{-1}y_0-\frac{3}{y_0}\left(1-\frac{1}{y_0}\tan^{-1}y_0\right)\right]\right\},\quad(17.1)$$

and

$$g_2{}^V(2\pi)\approx\frac{ef^2}{m_\pi{}^2}\left(\frac{M^3q_\pi{}^2}{mq_n{}^3}\right)$$
$$\times\left\{\tan^{-1}y_0-\frac{3}{y_0}\left(1-\frac{1}{y_0}\tan^{-1}y_0\right)\right\},\quad(17.2)$$

where now

$$y_0=\frac{2q_\pi q_n}{\frac{1}{2}m^2-m_\pi{}^2}.\quad(17.3)$$

One is tempted to assume that the weight functions $g_{1,2}(m^2)$ are everywhere reasonably well represented by this approximation and proceed to an evaluation of the structure factors. The anomalous (vector) nucleon magnetic moment obtained from Eq. (17.1) is $1.5e/2M$, quite near the experimental value $1.84e/2M$, although the close agreement must be fortuitous because the mean square radius of the magnetic moment, similarly calculated, is only about half the experimental value. Nevertheless we may regard the perturbation result as giving a qualitative and perhaps a semiquantitative representation of $g_2(m^2)$.

Assuming the same to be true for $g_1(m^2)$, one may use Eq. (17.2) to estimate the mean square radius of the vector-charge cloud [Eq. (4.2)]. The result for $(r_\rho{}^{-2})^V$ is $0.24m_\pi{}^{-2}$, which agrees with the measured value[4,6] within the fairly large experimental uncertainties. It should be remarked that these results for the vector charge and magnetic-moment structure obtained from the local theory, using only the Born approximation to the meson-nucleon scattering amplitude, are not very different from those given by the cutoff model in the same approximation (both being in reasonable agreement with experiment). That is to say, the effect of nucleon recoil in the Born contribution introduces a natural "cutoff" in the neighborhood of $m^2=(2M)^2$. Presumably, if a correct method for handling the scattering corrections could be formulated, a natural cutoff would appear there also.

VI. SUMMARY AND DISCUSSION

18.—The reader may at this point feel that the authors have perpetrated a fraud, cloaking nothing more than old-fashioned perturbation theory in a vast cloud of words and equations. To refute this impression let us review what has been accomplished, starting with the problem of the magnetic-moment structure, which is much clearer than that of the charge.

We began with the observation that in the framework of the spectral representation the observed qualitative properties of the anomalous nucleon magnetic moment suggest that it is due principally to the two-pion intermediate state. We then attempted a calculation of this contribution and had to deal with the problem of extending the meson-nucleon scattering amplitude into the region of negative squared momentum transfer. However, it was found that for small values of $m^2=-q^2$

the main part of the weight function $g_{2(2\pi)}{}^V(m^2)$ was due to the nucleon pole in the pion-nucleon scattering amplitude, which depends only on the renormalized Yukawa coupling constant and which can be extended without difficulty. Thus it seems reasonable to ignore the scattering corrections and to use only the nucleon pole in order to gain a rough idea of the content of the local theory. When this is done, one finds a magnitude for the static anomalous moment and a "size" which are in semiquantitative agreement with the observations. Our conclusion from this result is that a correct calculation based on the local theory may very well yield complete agreement with experiment. The fact that the practical estimate finally carried out here is equivalent to a piece of lowest-order perturbation theory is irrelevant to the validity of this estimate.

It has of course not been shown that more complicated intermediate states fail to contribute appreciably to the magnetic moment. Here we are unable even to make an estimate until some understanding has been developed of the matrix elements coupling these states on one side to the electromagnetic field and on the other side to the nucleon.

The particular high-mass intermediate state that has discredited local-field theory in the magnetic moment problem is the $N\bar{N}$ system, whose contribution when evaluated by perturbation theory is of the same order of magnitude as that of the two-pion state and which contains a large incorrect isotopic scalar part. The reader may well ask why he should disbelieve perturbation theory for the $N\bar{N}$ state when he is asked to accept it for the 2π configuration. The situations in these two cases, however, are quite different, because in the former the relevant scattering matrix element $(N+\bar{N}\rightarrow N+\bar{N})$ is to be evaluated in the physical region and there is no reason to think it is even remotely approximated by the second-order Born approximation. This approximation is known to be totally misleading even for nucleon-nucleon scattering, and in the nucleon-antinucleon problem the influence of annihilation processes on elastic scattering is enormous. Furthermore the $N\bar{N}$ contribution to the magnetic moment involves the nucleon structure factors $G_{1,2}{}^{S,V}$, which we *know* are important but which are ignored in the perturbation calculation. The corresponding pion-structure factor F_π, which occurs in the 2π contribution, *may* be important but there is no evidence to this effect.

19.—The situation with regard to the charge structure of the nucleon is not nearly so clear, but we feel that in this case also one should not conclude that local-field theory is incapable of ever explaining the known facts. In particular the fairly large "charge radius" observed for the proton[4] does *not* imply that the two-pion state is the main contributor. It is quite possible that an isotopic scalar part, approximately equal in magnitude to the vector part, will be forthcoming from the 3π state to produce the required small charge radius for the neutron.

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